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On locally biholomorphic finitely valent mappings from multi-connected domains onto the open disc ☆,☆☆

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ABSTRACT

In this paper we offer new results of research presented in the referred papers of J.E. Fornaess and E.L. Stout, E. Ligocka and the authors, and concerning the existence of m-valent locally biholomorphic mappings from product domains of $\mathbb{C}^n$ onto n-dimensional complex manifolds. In particular, we confirm an own conjecture about the estimation of the valentness m of locally biholomorphic mappings from multi-connected domains onto the open unit disc.

1. Introduction

When starting the studies of biholomorphic mappings in $\mathbb{C}^n$, we immediately meet a serious obstacle: a biholomorphic mapping of the open unit polydisc Δ^n onto the open Euclidean ball $\mathbb{B}^n$ does not exist (Poincare Theorem [9]). The obstacle disappears when we include locally biholomorphic mappings into our considerations. For instance, Fornaess and Stout [2] have proved that for every n-dimensional paracompact complex manifold Y there exists a locally biholomorphic and m-valent, $m \leqslant 2 + (2n+1)4^n$, mapping f from the open unit polydisc Δ^n onto Y, hence also onto the ball $\mathbb{B}^n$. Note that m-valence, $m \in \mathbb{N}$, of a mapping f from X onto Y means that for every $y \in Y$ the fibre $\{f^{-1}(y)\}$ has no more than m elements.

In 2003 Ligocka [8] replaced the polydisc Δ^n in Fornaess–Stout theorem by a Cartesian product $D_1 \times \cdots \times D_n$, of multi-connected domains D_j, $j = 1, \ldots, n$, but at a cost of worse estimation of valentness: $m \leqslant (24)^n[2 + (2n+1)4^n]$. This result follows from her theorem that every domain $D \subset \mathbb{C}$, such that $\overline{\mathbb{C}} \setminus D$ contains an isolated component not equal to a single point, can be mapped onto the open unit disc Δ locally biholomorphically and 24-valently.

In papers [6,7] and [10] we have considered the problem of decreasing the constant 24 for a wider class of domains (domains with isolated boundary fragments).

Definition 1. (See [6].) We shall say that a domain D of the complex plane $\mathbb{C}$ has an isolated boundary fragment, if at least one of the following conditions is fulfilled:

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(I) There exists a continuum $K \subset \partial D$ (a closed connected set including more than one point), and an open set U such that $K \subset U$ and $(\partial D \setminus K) \cap U = \emptyset$.
(II) There exists a Jordan arc $\Gamma \subset \partial D$ with distinct ends ξ, η and an open disc B such that $\xi, \eta \in \partial B$, $\Gamma \setminus \{\xi, \eta\} \subset B$ and $(\partial D \setminus \Gamma) \cap B = \emptyset$.
(III) There exists a point $a \in \partial D$ and an open disc $B(a)$ with centre at a, such that $(B(a) \setminus \{a\}) \cap \partial D = \emptyset$ (a is an isolated point of the boundary ∂D).

The above continuum K, Jordan arc Γ and point a will be called an isolated boundary fragment of D of type (I), (II), (III), respectively.

Of course, finitely connected domains are domains with isolated boundary fragments.

There exist domains with isolated boundary fragment of third type, those cannot be mapped onto Δ locally biholomorphically and finitely valently. An example of that domain is $\mathbb{C} \setminus \{0\}$, because each such mapping has an isolated singularity $z = 0$, which is impossible.

However, for the domains with an isolated boundary fragment of type (I) and (II), in [6] we have proved:

Theorem A. *(See [6].) If $D \subset \mathbb{C}$ is a multi-connected domain, with an isolated boundary fragment of type* (I) *or type* (II)*, then there exists a locally biholomorphic* 5*-valent mapping from D onto the disc Δ.*

This generalizes the above cited Ligocka's result, because the class of domains D with isolated boundary fragments of type (I) and (II) is wider than the class of domains D with isolated component of $\overline{\mathbb{C}} \setminus D$ and our estimation $m \leqslant 5$ is better than Ligocka's estimation $m \leqslant 24$.

In the same paper we have also given the following conjecture:

Conjecture. *(See [6].) The constant* 5 *from Theorem A can be replaced by the constant* 3.

In connection with this conjecture in [7] we have proved the inequality $m \leqslant 4$. On the other hand, the second author has shown [10] that the constant $m = 5$ cannot be replaced by a constant $m < 3$, which means that there exist domains D with an isolated boundary fragment of type (I) and also of type (II) that cannot be mapped onto Δ locally biholomorphically and less than 3-valently. From this and from the main theorem of the present paper it follows that the above conjecture is true. Therefore, every domain D with isolated boundary fragments of type (I) and (II) can be mapped onto the disc Δ locally biholomorphically and 3-valently and the constant 3 cannot be decreased in the considered class of domains.

2. New results

Theorem 1. *If $D \subset \mathbb{C}$ is a domain with an isolated boundary fragment of type* (I) *or* (II)*, then there exists a* 3*-valent, locally biholomorphic mapping from D onto the disc Δ.*

Proof. (1) Let us assume that D has an isolated boundary fragment of the first type and let K be the same continuum as in the definition. After using a homography we can assume that $\infty \in D$. Then there exists a simply connected domain D_0 such that $D \subset D_0$ and $\partial D_0 = K$. It is obvious if $\overline{\mathbb{C}} \setminus K$ is connected and in the opposite case we can use a decomposition theorem (see [5, Chapter V]).

From the definition it also follows that there exists a compact $K_0 \subset D_0$ such that $\partial D \setminus K \subset K_0$. Let Φ be a biholomorphic mapping from the domain D_0 onto unit disc Δ and let $K^0 = \Phi(K_0)$. We can assume that there exists a circular open neighborhood B' of the point $\zeta = -1$, such that $K^0 \cap B' = \emptyset$.

Let $\Pi^+ = \{z \in \mathbb{C}\colon\ \operatorname{Im} z > 0\}$ and let Π^+_* be the simply connected domain that we obtain from Π^+ by removing two vertical segments $(-\sqrt[4]{\frac{3}{4}}, \sqrt[4]{3}e^{i\frac{3}{4}\pi}]$, $(\sqrt[4]{\frac{3}{4}}, \sqrt[4]{3}e^{i\frac{\pi}{4}}]$.

Let Ψ be a biholomorphic mapping from Π^+_* onto Δ and let $E = \Psi(B^+_r)$, where $B^+_r = \{z \in \Pi^+\colon\ |z| < r\}$ and r is a fixed number from the interval $(0, \sqrt[4]{\frac{3}{4}})$. The mapping Ψ can be extended to a homeomorphism from $\Pi^+_* \cup [-r, r]$ onto $\Delta \cup S$, where $S \subset \partial\Delta$ is a Jordan arc (see [4, Chapter II, §3]). Let us also denote the extended mapping by Ψ. We can assume that the point $\zeta = 1$ belongs to the arc $S = \Psi([-r, r])$ and it is different from the ends of S.

Then there exists a circular open neighborhood B'' of the point $\zeta = 1$, such that $B'' \cap \Delta \subset E$. Supposing the contrary, we can find an infinite sequence $(\zeta_\nu) \subset \Delta \setminus E$ convergent to the point $\zeta = 1$. Then the sequence $z_\nu = \Psi^{-1}(\zeta_\nu)$, $\nu \in \mathbb{N}$ converges to the point $z_0 = \Psi^{-1}(1) \in (-r, r)$. On the other hand $z_\nu \in \overline{\Pi^+_* \setminus B^+_r}$, because for every $\nu \in \mathbb{N}$ $z_\nu \in \Pi^+_*$ and $z_\nu \notin B^+_r$. Hence, $z_0 \in \overline{\Pi^+_* \setminus B^+_r}$, which is impossible, because $\overline{\Pi^+_* \setminus B^+_r} \cap (-r, r) = \emptyset$.

The mapping

$$\Theta(z) = \frac{z + a}{1 + az}, \quad z \in \Delta,$$

with parameter $a \in (-1, 1)$ sufficiently close to 1, transforms the domain $\Delta \setminus \overline{B'}$ into the domain $\Delta \cap B''$.

Then the mapping $\Psi^{-1} \circ \Theta \circ \Phi$ transforms D biholomorphically onto a domain $D_1 \subset \Pi_*^+$ such that $\partial D_1 \setminus \partial \Pi_*^+ \subset B_r^+$.

Now let us consider the function

$$f(z) = \frac{z^3 - 3z}{z^2 - 1}.$$

In [6] we have proved that f is 3-valent in the open upper half-plane Π^+ and $f(\Omega) = \Pi^+$, where

$$\Omega = \left\{z \in \Pi^+ \colon |z| > \sqrt[4]{3}\right\} \cup \left\{\sqrt[4]{3}e^{it} \colon t \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{3}{4}\pi, \pi\right)\right\} \subset D_1.$$

Since $f'(z) = 0$ iff $z = \sqrt[4]{3}e^{i\frac{\pi}{4}}$ or $z = \sqrt[4]{3}e^{i\frac{3}{4}\pi}$, the function f is locally biholomorphic in D_1.

Therefore, the function $F = f \circ \Psi^{-1} \circ \Theta \circ \Phi$ maps domain D onto the upper half-plane Π^+ locally biholomorphically and 3-valently.

It remains to map Π^+ biholomorphically onto disc Δ.

(2) Let us assume that D has an isolated boundary fragment of the second kind.

Let Γ, B be the same as in the definition, and $\gamma \subset D$ be an open arc of the circle ∂B, joining the ends of Γ. Let G be this subdomain of the domain D that is included between arcs Γ and γ.

By K let us denote a component of the boundary ∂D such that $\Gamma \subset K$ and by D_0, similarly as in the first part of the proof, let us denote a simply connected domain such that $D \subset D_0$, $\partial D_0 = K$. In view of Riemann theorem there exists a biholomorphic mapping g from D_0 onto the upper half-plane Π^+. By D_*, G_*, γ_* let us denote the images, in the above mapping, of the sets D, G, γ, respectively.

Since $\partial G = \gamma \cup \Gamma$ is a Jordan curve, all points of the boundary ∂G are attainable from the domain G. Of course, γ_* is the homeomorphic image of γ and all points of γ_* are attainable from the domain G_*. Since the set of points that correspond to points of ∂G_* available from G_*, is dense everywhere in $\gamma \cup \Gamma$ (see [4, Chapter II, §3]), then there exist at least two points $\zeta_1, \zeta_2 \in \partial G_* \setminus \gamma_*$, available from the domain G_*. From this it follows that $\zeta_1, \zeta_2 \in \mathbb{R}$.

Let $T_1, T_2 \subset G_*$ be two disjoint Jordan arcs, with ends in ζ_1, ζ_2, respectively. Then there exists a Jordan arc $T_3 \subset G_*$ that joins other ends of T_1, T_2 in such a way that $T_1 \cup T_2 \cup T_3$ will also be a Jordan arc.

Thus, in a domain $G'_* \subset G_*$ bounded by the Jordan curve $T_1 \cup T_2 \cup T_3 \cup [\zeta_1, \zeta_2]$, there exists a semi-disc with the centre $z_0 \in \mathbb{R}$ and the diameter included in $\mathbb{R}$. Note that $g((\partial D) \setminus K)$ is situated out of G'_*.

A biholomorphic mapping from Π^+ onto itself, sending z_0 onto ∞, transforms D_* onto a domain $\widetilde{D} \subset \Pi^+$ such that $\partial\widetilde{D} \setminus \mathbb{R}$ will be included in a disc centered at zero, of a sufficiently large radius.

Now we map the upper half plane Π^+ (including $\widetilde{D}$) onto the disc Δ by a homography that sends ∞ to -1. Then the domain $\widetilde{D}$ will be transformed onto a domain $\widehat{D} \subset \Delta$ that includes the domain $\Delta \cap B'$, where B' is a circular open neighborhood of the point $\zeta = -1$ (as in the first part of the proof). Next it is sufficient to use similar steps as have been used there. □

Let us observe that if we join our Theorem 1 and the previously cited Fornaess–Stout result [2], we obtain the following corollaries:

Corollary 1. *If a multi-connected domain $D \subset \mathbb{C}$ has an isolated boundary fragment of the first or second kind and X is an open connected Riemann surface, then there exists $m \in \mathbb{N}$ and m-valent locally biholomorphic mapping f from D onto X and $m \leqslant 3 \times 14 = 42$.*

Corollary 2. *If $X = D_1 \times \cdots \times D_n$, where domains D_j, $j = 1, \ldots, n$, fulfill the assumptions of Corollary 1, and Y is a connected paracompact n-dimensional complex manifold, then there exists a locally biholomorphic and m-valent mapping f from domain X onto manifold Y and $m \leqslant 3^n[(2n+1)4^n + 2]$.*

In supplement to the above corollaries note that in paper [3] there is information that in Bushnell paper [1] the constant $(2n+1)4^n + 2$ is replaced by the constant $(2n+1)^2 + 1$.

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